A Markovian Exploration of Monopoly

Chris Gartland, Hannah Burson, and Tim Ferguson

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0.1 Introduction

In this paper, we investigate the classical board game Monopoly using basic mathematical techniques. We do this by modeling the probabilities of players landing on any given space, which we show varies from space to space. As a result, it is possible for a property with lower rent to make more money than another property with higher rent. An example of this is that a hotel on Pennsylvania Avenue is worth more than a hotel on Park Place. In our analysis, we consider questions such as "When are two smaller monopolies worth a single larger one?", "How much are utilities and railroads worth?", and "Which properies should I develope first?". These results will help a player trade and develope properties with greater returns.

0.2 Monopoly as a Markov Chain

We model the probability of ending a turn on a given monopoly space as a Markov chain. This means that the probability of ending a turn on a space depends only on the probabilities of ending the previous turn on the other spaces and not on any earlier history. We construct a matrix M where entry M_{ij} is the probability of moving from state j to state i at the end of the next turn. The matrix M is a Markov matrix, and thus it has an eigenvalue of 1 occuring with multiplicity 1, and all its other eigenvalues are less than 1 in modulus. The appropriately normalized eigenvector associated with the eigenvalue 1 then represents the steady-state probabilities of ending a turn on the monopoly spaces. We use these steady-state probabilities for decision-making analysis in monopoly game play.

0.3 Code Implementation

We created our matrix to store more information than just the spaces on the board. There are three important pieces of information we needed to store:

1. When a player rolls doubles three times in a row, she is sent to jail. Thus, we split each space into three lines in our matrix. The first line represents no doubles, the second line represents one roll of doubles, and the third line represents the second roll of doubles.

2. One of the cards in the Chance deck requires the player to move back three spaces. One of the Chance spaces is three spots ahead of a Community Chest space. We used six lines to represent this community chest spot, three for landing on the space with a roll and three for landing on this space form moving from chance. We will show the importance of the extra lines when we explain how we created our matrix.

3. As the game progresses and players start owning monopolies, a common strategy is to stay in jail as long as possible. To implement the rules of jail

(roll each turn, leave jail if you roll doubles, pay to leave after the third failed attempt at rolling doubles), we created three lines for the in jail space. On each turn, the player rolls. If no doubles, she would be stored in the next line of jail until their are no lines left and she must move the number shown on the dice. These three lines are seperate from the three lines representing the just visiting part of the jail space.

We also created a similar matrix using the strategy of immediately paying to get out of jail. This strategy is common at the beginning of the game as players try to aquire as much property as possible.

We created our matrix by splitting the roll into three parts and creating one matrix for each part. The first matrix (M_1) represents the movement based on the roll. The number in space M_{ij} represents the probabily of rolling the dice in a way that you land on space i if you were on space j. In our code, we used a function, roll, which is given the number of the line representing the space the player is initially on and two numbers that represent the roll of the dice. In the function, we move the player the amount shown in the sum, test for doubles, and adjust the doubles count. We also include the attempts to move out of jail in M_1.

The second matrix (M_2) performs the transformation that considers the special roles certain spaces play. For example, M_2 sends any player on the "Go to Jail" space to Jail. Additionally, of the 16 possible chance cards, nine of them send you to other spaces and of the 16 community chest cards, two of them send you to other spaces.

The third matrix (M_3) is very sparse. It considers the one transition left. If a player rolls to land on the community chest on the side between Go to Jail and Go, the probability of her drawing a card and moving to another square is calculated in the second matrix. However, a player who lands on the Chance space on that same side could draw a card that moves her back three spaces to the Community Chest space. That move is done by the second matrix. But, the player will draw a card from community chest when she moves to that spot. Thus, the third matrix does the transformation from that final draw of community chest.

The transition matrix M for our Markov chain is $M_3 * M_2 * M_1$.

0.4 Analysis

Since the matrix M is a Markov matrix, meaning all its entries are nonnegative and its columns sum to 1, M has a single eigenvalue of 1 and the rest of its eigenvalues have modulus less than 1. We normalized the eigenvector associated to 1 so the sum of its entries is 1. This normalized eigenvector then represents the steady-state probabilities of ending a turn on our designated states. These results are shown in Figure 1. Furthermore, we computed the modulus of the second largest eigenvalue to be about 0.87, so probabilities converged to the steady-state probabilities on the order of $O(0.87^n)$.

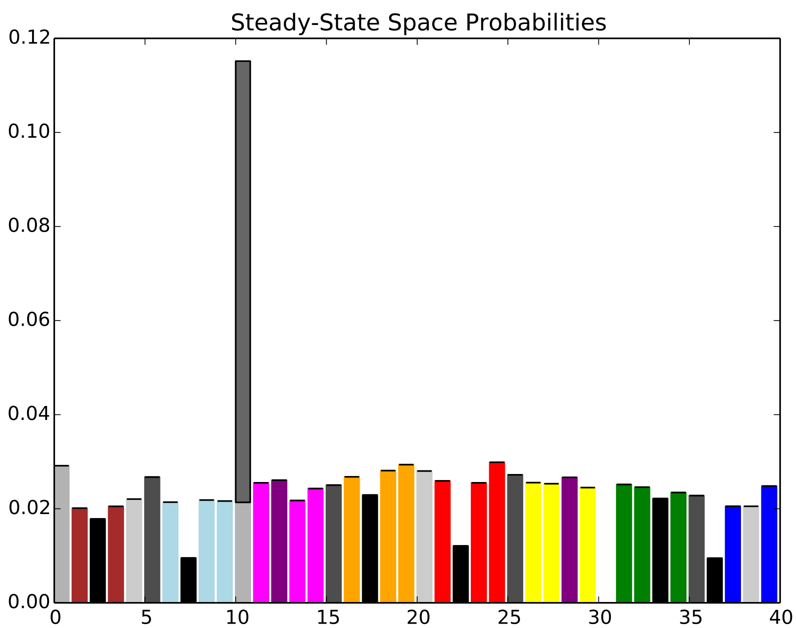

Figure 1: Steady-State Probabilities of Ending a Turn on a Space

As explained in Section 0.3, each space has three states associated to it, except the last community chest, which has six states. We sum the states for each space together to obtain the probability of ending a turn on the space. This information allows us to calculate the expected earnings from rent paid on properties owned. For example, the steady-state probability of ending a turn on Park Place is 2.05%, and the steady-state probability of ending a turn on Boardwalk is 2.48%. With hotels erected on Park Place and Boardwalk, the rents owed for landing on the spaces is $1500 and $2000, respectively. Then for a player owning the blue monopoly with hotels erected, the expected dollars earned per opponent roll is $1500 \cdot .0205 + 2000 \cdot .0248 = 80.43$. Similar earnings calculations were performed for a player owning the following properties:

1. Any monopoly with three houses erected
2. Any monopoly with hotels erected
3. 1, 2, 3, or 4 railroads
4. Any combination of utilities

The results of these combinations are summarized in Tables 2, 1, 3, and 4. This information is the most useful since the rent on an unmonopolized property is negligible compared to a monopolized one with houses or to multiple railroads. The 3-house and hotel development information was included for the following reasons: In the early stages of a monopoly development, the additional rent gained by adding a house increases until three houses are erected. Thus, a 3-house develpoment is the most cost efficient. In the late stages of a game, monopolies have typically been fully developed, and thus hotel-developed properties are common.

Table 1: Monopoly Values with Three Houses

Monopoly Color Owned	expected dollars earned/opponent roll	
	calculated probability	uniform propability
brown	5.50	6.75
light blue	18.17	21.00
pink	33.41	35.00
orange	47.81	42.50
red	58.41	53.75
yellow	61.53	61.25
green	68.22	70.00
blue	57.33	62.50

Table 2: Monopoly Values with Hotels

Monopoly Color Owned	expected dollars earned/opponent roll	
	calculated probability	uniform propability
brown	14.26	17.50
light blue	36.77	42.50
pink	57.31	60.00
orange	81.52	72.50
red	86.87	80.00
yellow	87.91	87.50
green	96.25	98.75
blue	80.43	87.50

Table 3: Railroad Values

Number of Railroads Owned	expected dollars earned/opponent roll	
	calculated probability	uniform propability
1	0.64	0.63
2	2.54	2.50
3	7.63	7.50
4	20.35	20.00

Table 4: Utility Values

Utility Owned	expected dollars earned/opponent roll	
	calculated probability	uniform propability
Electric	0.61	0.70
Water Works	0.71	0.70
Both	3.31	3.50

These tables can be used to estimate the long term effects of different trades. For example, a common trade made in monopoly results in one player getting a large monopoly and another player getting two smaller ones. A specific example is that one player gets the green monopoly and the other player gets the yellow and brown monopolies. The question is "Does including the brown monopoly into the deal make up for the difference between the green and yellow monopolies?". Using the table for monopolies with three houses we see that the expected return is 68.22 dollars per roll for the green monopoly while the expected return is $61.53 + 5.50 = 67.03$ dollars per roll for the yellow and brown monopolies. In this case the brown monopoly doesn't make up the difference. However, if we changed green to yellow and yellow to red in the trade, the expected return is 61.53 dollars per roll for the yellow monopoly and 58.41 + 5.50 = 63.91 dollars per roll for the red and brown monopolies. Now the brown monopoly more than compensates for the difference. Note that if we had used the uniform probability, the single large monopoly would have a higher expected return than the two smaller monopolies in both cases.

An example involving railroads is a trade that results in one player getting a third railroad and another player getting the brown monopoly. The increase in the expected return from the railroads is 7.63 - 2.54 = 5.09 dollars per roll where as the expected return from the brown monopoly with three houses is 5.50 dollars per roll. If we change the deal to include one more railroad and change brown to light blue, then the increase in the expected return from the railroads is 20.35 - 2.54 = 17.81 dollars per roll where as the expected return from the light blue monopoly is 18.17 dollars per roll. Of course, railroads don't have to be developed, and as a result, getting the railroads may be the better end of the deal in these cases, or the player getting the railroads should have to pay some amount of money to the other player. For example, according to Wikipedia the average length of a monopoly game is 1 - 4 hours. Assuming that it takes about a minute for each turn, the player getting the monopoly will make at most approximately 240 $\times$ (18.17 - 17.81) = 86.40 more than the other player over the course of the game. However, it costs 450 dollars to build develope the light blue monopoly with three houses on each space. Therefore, the player getting the monopoly should ask for at least 450 - 86.40 = 363.60 dollars in the deal. Note that if we had used the uniform probability, the results would have been the same but with a larger margin.

Generally speaking, the utilities are not worth much, but can be considered in trades the same way.

Our probability analysis can also be used to help a player decide where to erect houses if she has a fixed amount of money. For example, suppose a player owns the green and blue monopolies with two houses erected on Park Place, three houses on Boardwalk, and two houses erected on each of the green properties. Suppose also that the player only has 200 dollars to spend, and she wishes to erect a third houses on either Park Place or Pennsylvania Avenue. The cost of a house is 200 dollars for each of these properties, but the increase in rent is 550 dollars for Pennsylvania Avenue and 600 for Park Place. If the player assumed equal probabilities of ending a turn on these spaces, she would expect that erecting the house on Park Place would be more profitable. However, as shown in Table 5, the probability of ending a turn on Pennsylvania Avenue is 2.34% compared to 2.05% for Park Place. The expected earnings increase is then $550 \cdot .0234 = 12.89$ dollars/opponent roll for adding a third house to Pennsylvania Avenue but only $600 \cdot .0205 = 12.30$ dollars/opponent roll for adding a third house to Park Place.

We have focused our analysis on the case later in the game when players stay in jail as long as possible. According to http://www.amnesta.net/other/monopoly/, a good strategy is to start staying in jail once other players have monopolies with houses. Most of the questions we are focusing on occur once players have monopolies. However, players will make choices about trades while still trying to get out of jail quickly to collect property. Thus we can consider the trade involving one player gaining the third railroad while the other player gains the brown monopoly. The expected gain in return for the third railroad is $\$8.19 - \$2.73 = \$5.46$ while the brown monopoly with three houses has an expected return of \$5.84 per roll. These amounts are fairly comparable, especially because the player with the brown monopoly has to pay to build the houses. Note that when we compare the same calculations with the two strategies, both have the brown monopoly gaining slightly more money on each roll, but the total amount of money for each roll in the case of staying in jail is slightly less.

Table 5: Pennsylvania Avenue and Park Place Hotel Cost, Rent Change, and Probabilities

Property	House Cost (dollars)	Rent Increase from 2 to 3 Houses (dollars)	Steady-State Probability
Pennsylvania Avenue	200	550	2.34%
Park Place	200	600	2.05%