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July 25, 2014

1 Introduction

Given a graph G, we place an oscillator at each node i that that operates at a natural frequency of ω_i and has a phase of θ_i. The oscillators can interact with each other nonlinearly using the interaction function

$$\gamma_{ij} f(\theta_i - \theta_j),$$

where f is some odd periodic function and γ_{ij} is the strength of the interaction. This leads to the system of differential equations

$$\frac{d\theta_i}{dt} = \omega_i + \sum_j \gamma_{ij} f(\theta_i - \theta_j).$$

We assume that $\gamma_{ij} = 1$ and $f(x) = \sin x$. Without loss of generality, we may assume that

$$\sum_i \omega_i = 0$$

by moving our system to a rotating coordinate system. Then for certain configurations of initial ω_i's, the system will synchronize such that the θ_i's are eventually constant. So in this case, the θ_i's are fixed points of the system, so $\frac{d\theta_i}{dt} = 0$. Now suppose that we represent the system of differential equations as

$$\frac{d\vec{\theta}}{dt} = \vec{g}(\vec{\theta}),$$

and that $\vec{\theta}^*$ is a fixed point of the system. Then if $\vec{\theta} = \vec{\theta}^* + \epsilon\vec{y}$, we can linearize to find that

$$\frac{d\vec{y}}{dt} = \vec{\nabla}\vec{g}(\vec{\theta}^*) \cdot \vec{y}.$$

Thus if the eigenvalues of $\vec{\nabla}\vec{g}(\vec{\theta}^*)$ are negative, this will tend toward $\vec{0}$, giving us the desired stability. To evaluate the Jacobian, we find that

$$\frac{\partial g_i}{\partial \theta_j} = \begin{cases} \gamma_{ij}\cos(\theta_j - \theta_i) & \text{if } i \neq j \\ -\sum_k \gamma_{ik}\cos(\theta_k - \theta_i) & \text{if } i = j. \end{cases} \tag{1}$$

So if $\gamma_{ij} \geq 0$ and $|\theta_j - \theta_i| \leq \frac{\pi}{2}$, then the Jacobian will be like the graph Laplacian.

In this project, we investigated the stability of specific graphs given natural frequencies ω_i. Using the ODE solver `odeint` from Python, we found which natural frequencies lead to the synchronization of phases, i.e. $\vec{\theta}$ converging to some vector. To determine if a configuration synchronized, we found the difference between the $\vec{\theta}$'s far enough out in time that the configuration would have converged. If the norm of this difference was small (less than .1), we said that the configuration synchronized. In addition, we examined the behavior when we changed f or the topology of the graph G.

2 Synchronization

One of the questions we explored was which values of ω result in synchronization. We were able to explore this question on the graph K_3. Because the ω_is must sum to zero, $\omega_3 = -\omega_1 - \omega_2$. Thus, we can create a two dimensional plot with ω_1 as the x-coordinate and ω_2 as the y-coordinate.

We first focused on $f(x) = \sin x$. For this function, the points where the system synchronizes lie in an ellipse (pictured in 1) with major axis $\sqrt{6}$, minor axis $\sqrt{2}$, and a rotation of $45°$ counter-clockwise.

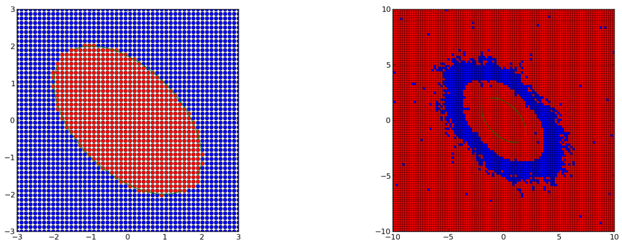

Figure 1: Points of synchronization for $f(x) = \sin x$ and $f(x) = \sin(3x)+\sin(5x)$

We then explored $f(x) = (\sin x)^3$. The points where this system synchronizes lie in roughly the same space covered by the ellipse in the previous example, but have a hexagonal shape as shown below. Similarly, the function $f(x) = (\sin x)^5$ follows a similar pattern, covering slightly less space than the hexagonal shape above, this time with concave edges.

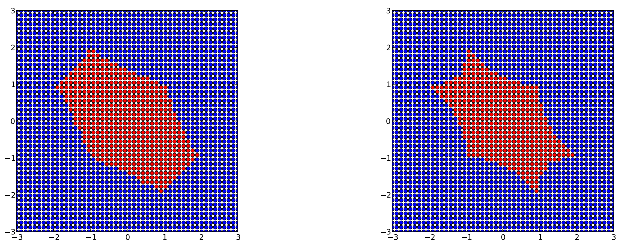

Figure 2: Points of synchronization for $f(x) = \sin^3 x$ and $f(x) = \sin^5 x$

Another group of functions that we examined consisted of several sums of sines. This included $f(x) = \sin(3x) + \sin(5x)$, $f(x) = \sin(3x) + \sin(5x)$ and $f(x) = \sin(x/3) + \sin(3x)$. The points where these systems synchronize form variations on the ellipse generated by $f(x) = \sin x$.

One function that does not seem to follow any consistent pattern is $f(x) = \cot x$, as shown in 3 . The points where the cotangent function synchronizes

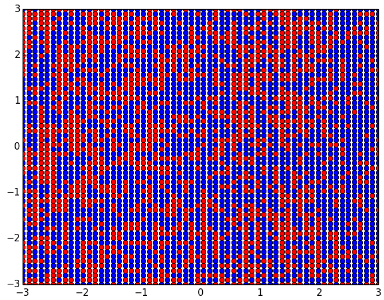

Figure 3: Points of synchronization for $f(x) = \cot x$

seem to be distributed more or less randomly throughout the plot. The cosecant function shares this characteristic. Depending on the random initial values, the function may or may not converge to a synchonized steady state.

We also looked at what would happen if we also allowed the γ_{ij}'s to vary, adding the equations

$$\frac{\gamma_{ij}}{dt}\cos(\theta_i - \theta_j) - \frac{1}{|E|}\sum_{i,j \text{ connected in } G}\cos(\theta_i - \theta_j)$$

to the system of differential equations, where $|E|$ is the number of edges in G. Using $f(x) = \sin x$, we found that the system synchronized for the points shown in red in figure 4.

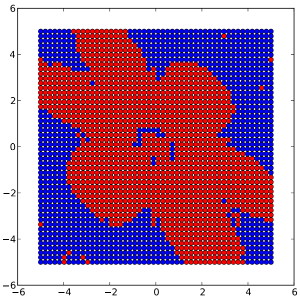

Figure 4: Points of synchronization for $f(x) = \sin x$ and γ interaction

These points appear to form an ellipse with six pieces jutting out symmetrically. To get a better picture of this, we can use the change of variables $u = (x - y)/\sqrt{3}$, $v = x + y$ to obtain the following picture. Included in green are a circle of radius 3 centered at $(0, 0)$ and six equally spaced circles of radius $\sqrt{2}$ centered 4 units away from $(0, 0)$.

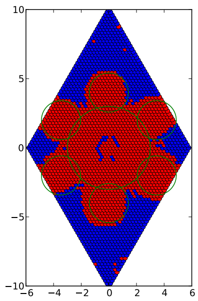

Figure 5: Points of synchronization for $f(x) = \sin x$ and γ interaction (translated)

3 Stable Synchronized States

Though we could not easily graph the ω_is that allowed for synchronization on larger graphs that K_3, we were still able to answer other questions about these graphs. Mostly, we focused on how many stable synchronized states we found.

Our first exploration used complete graphs and $f(x) = \sin x$. For this case, all the initial values for the θ_is, gave synchronized states that were the same up to translation. See figure 6 for the solution.

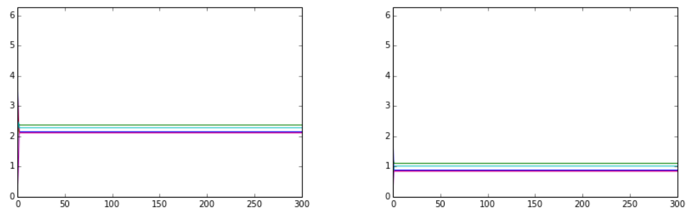

Figure 6: Steady-State Solutions for $f(x) = \sin x$ and K_5

We then explored cyclic graphs and $f(x) = \sin x$. In this case, we found more than one steady state solution. See figure 7 for the solutions.

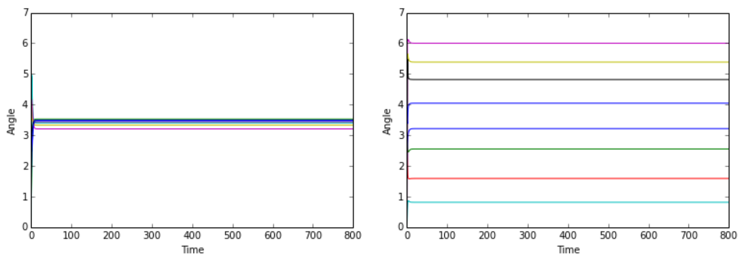

Figure 7: Two Steady-State Solutions for $f(x) = \sin x$ and C_8

We explored other odd, periodic functions and other graphs as well. One set of functions we considered was those of the form $\sin(ax) + \sin(bx)$. These functions resulted in more than one steady state for the complete, cyclic, and bipartite graphs we looked at. See figures 8, 9, and 10 for some examples.

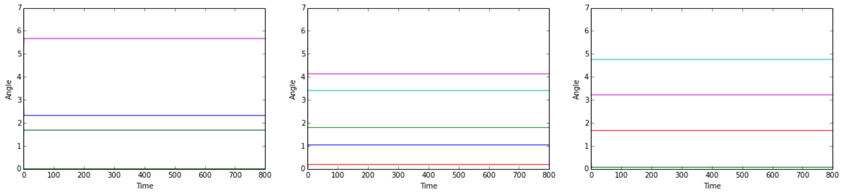

Figure 8: Three Steady-State Solutions for $f(x) = \sin 3x + \sin 8x$ and K_5

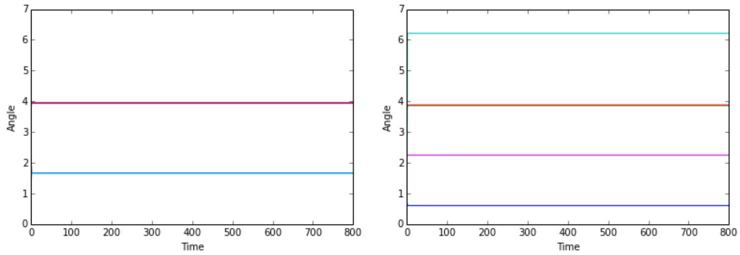

Figure 9: Two Steady-State Solutions for $f(x) = \sin 3x + \sin 8x$ and C_5

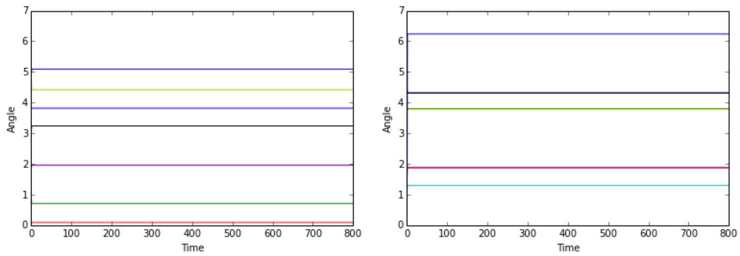

Figure 10: Two Steady-State Solutions for $f(x) = \sin 3x + \sin 8x$ and K_{35}

Another function we considered was $\sin^5(x)$. This function gave only one steady-state solution for all initial angles.

We also looked at other odd, periodic trig functions. An interesting example was $\csc x$. Depending on the initial value of the angles, the system may or may not reach a steady state. See figure 11 for examples on K_5.

Using equation (1), we can compute the Jacobian for a synchronized set of θ's. For example, we know that K_3 with natural frequencies $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ can synchronize to the phases $\theta_1 = \theta_2 = \theta_3$ (where the common value depends on the initial θ). In this case, we find that the eigenvalues of the Jacobian are

$$\lambda_1 = -3, \lambda_2 = 0, \lambda_3 = -3$$

This agrees with our assessment in the introduction, that the eigenvalues of the Jacobian for a configuration that synchronizes must not be positive.

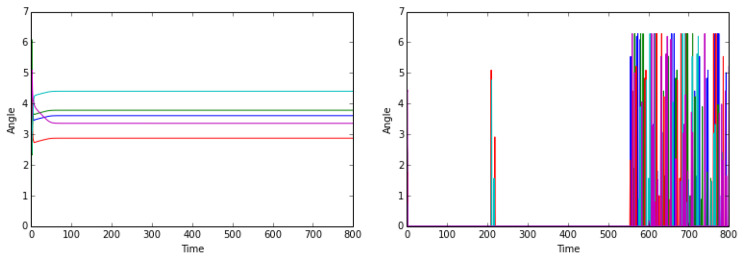

Figure 11: Two different solutions for $f(x) = \csc x$ and K_5